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Fractal Theory Space: Spacetime of Noninteger Dimensionality

Christopher T. Hill¹

¹*Fermi National Accelerator Laboratory
P.O. Box 500, Batavia, Illinois 60510, USA* *

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Abstract

We construct matter field theories in “theory space” that are fractal, and invariant under geometrical renormalization group (RG) transformations. We treat in detail complex scalars, and discuss issues related to fermions, chirality, and Yang-Mills gauge fields. In the continuum limit these models describe physics in a noninteger spatial dimension which appears above a RG invariant “compactification scale,” M . The energy distribution of KK modes above M is controlled by an exponent in a scaling relation of the vacuum energy (Coleman-Weinberg potential), and corresponds to the dimensionality. For truncated- s -simplex lattices with coordination number s the spacetime dimensionality is $1 + (3 + 2 \ln(s) / \ln(s + 2))$. The computations in theory space involve subtleties, owing to the $1+3$ kinetic terms, yet the resulting dimensionalities are equivalent to thermal spin systems. Physical implications are discussed.

*e-mail: hill@fnal.gov

1 Introduction

All quests for organizing principles of physics beyond the Standard Model, since the classic era of grand unification in the late 1970's, have involved extra dimensions. The foremost example is supersymmetry, [1], in which one postulates Grassmanian extra dimensions and graded extensions of the Lorentz group. Supersymmetry and bosonic extra dimensions are essential to the use of string theory with matter fields as a complete description of all forces, including quantum gravity. Motivated by certain viable limits of string theory, [2], the possibility of extra conventional spatial dimensions at the \sim TeV scale, possibly accessible to future colliders, has lately become the focus of a lot of activity. Latticization, [3], or “deconstruction,” [4], of an extra dimensional compactified theory provides an effective gauge invariant Lagrangian in $1 + 3$ dimensions truncated on N KK modes of scalars, fermions and gauge fields in D dimensions. This has provided a point of departure for abstracting a new class of models based upon the notion of “theory space” as emphasized by Arkani-Hamed, Cohen and Georgi [4].

Theory space, without some defining principles, is an empty concept. A key idea we emphasize presently is that theory space can be endowed with certain geometrical symmetries that are essentially renormalization group transformations. In the present paper we study a nontrivial example. In particular, we will borrow from condensed matter physics certain recursively defined, or *fractal* lattices to construct classes of new theory spaces. These lattices are defined by recursively “decorating” a kernel lattice of coordination number, s , by replacing each site with a simplex of n sites, preserving the coordination number s . This process is iterated an arbitrary number, k , times. It results in a “fractal” lattice, which for us describes a fractal theory space matter field theory.

The Feynman path integral is then found to be invariant under a sequence of renormalization group (RG) transformations that map the k th lattice into the $k - 1$ lattice. In the large k limit, this RG invariance implies a certain scaling property of the vacuum action functional, *e.g.*, the Coleman-Weinberg potential. This scaling behavior, and consistency with the RG symmetry, leads to the determination of an exponent associated with the number distribution of KK modes with energy:

$$N(E) \propto \left(\frac{E}{M}\right)^\epsilon \quad (1.1)$$

Here ϵ is the dimensionality of the extra dimensions; M is an effective “compactification scale” invariant under RG transformations. The effects of the extra dimension show up only for energy scales $E \geq M$ as KK-modes sporadically appear. We will obtain

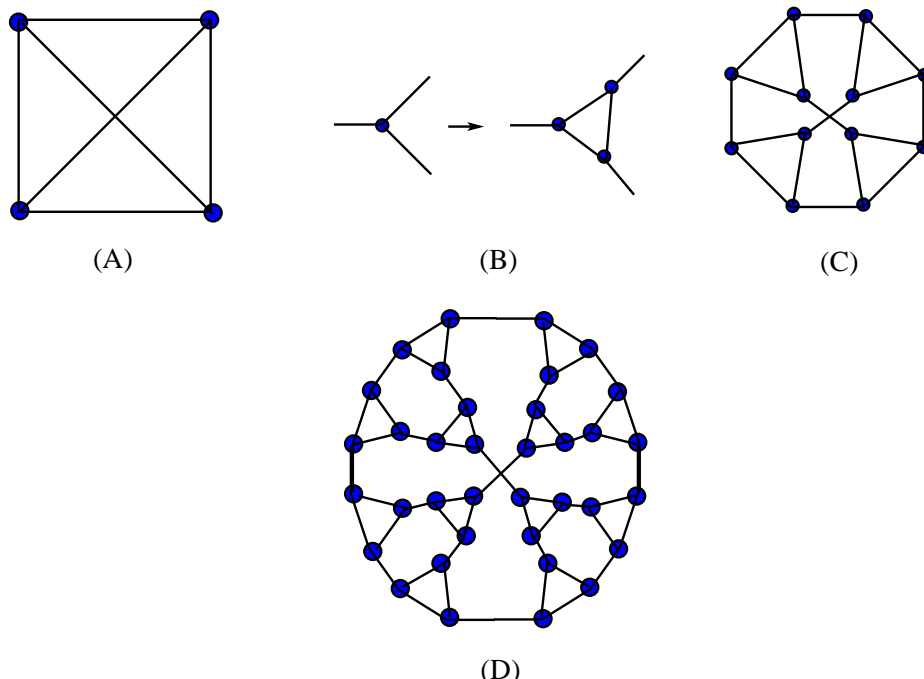


Figure 1: The truncated 3-simplex lattice. (A) Kernel (complete) lattice with coordination number 3; (B) the decoration which replaces each site under recursion; (C) the first order decorated lattice; (D) the second order decorated lattice. A theory space can be constructed by defining each site to correspond to a complex scalar $Z|\partial\phi_a|^2 - \mu^2|\phi_a|^2$, and each link to $-\Lambda^2|\phi_a - \phi_b|^2$. On the k th order truncated s -simplex we have the number of sites $N_k = (s+1)s^k$, and number of links, $L_k = (s+1)s^{k+1}/2$.

irrational values for ϵ for the geometrical RG transformations considered presently. Since theory space, endowed with such a geometrical symmetry, is effectively dual to a theory of compact extra dimensions in the continuum limit, we have thus arrived at a prescription for constructing a spacetime field theory in noninteger dimensionality.

More specifically, the kernel lattices we consider are “complete” lattices in which every site (complex scalar field) is coupled through a hopping term to every other site. For example, in Fig.(1) we show the square kernel, as a zeroth order lattice with coordination number 3 (the truncated 3-simplex). We then construct the next order lattice by “decorating” each vertex with a simplex. In Fig.(1C) we have decorated the kernel with the 3-simplex to produce the first order lattice. We then iterate the decoration to produce the second order lattice of Fig.(1D). The procedure can be iterated k times, and we imagine $k \rightarrow \infty$ to define a continuum limit.

The renormalization group transformations that reduce the k th order lattice Lagrangian back to the $(k-1)$ th order lattice, preserving the Feynman path integral, are

typically a sequence of a polygon- \star transformations, as first discovered by Onsager for the Ising model [8], followed by 4-chain \rightarrow 2-chain dedecorations. These transformations are adapted to the $1+3$ field theories that live on sites of the theory space lattice. One obtains the effective Lagrangian in the $(k-1)$ th lattice, with parameters that are renormalized under the transformations. The consistency of the RG symmetry, *i.e.*, of the invariance of the Feynman path integral, is realized only for a particular value of ϵ .

Such RG manipulations are familiar from the condensed matter literature, but are tricky in theory space in a fundamental way: the deconstructed theory possesses continuum kinetic terms for the field theory in the $1+3$ Lagrangian. We must include renormalization effects on these kinetic terms, up to irrelevant operators that are quartic derivatives, e.g. $(\partial^2\phi)^2/\Lambda^2$. In particular, $|\partial(\phi_a - \phi_b)|^2$ must be interpreted as a quartic derivative. These irrelevant operators of the derivative expansion are dropped, and the renormalization of the relevant $|\partial\phi|^2$ terms is determined. This renormalization plays a crucial role in the scaling law for the Coleman-Weinberg potential.

The solution to the problem of extracting ϵ essentially adapts the scaling theory of critical exponents [5]. We follow closely the beautiful approach of Dhar [6], who also discussed many other lattices. Though the physical systems we consider are different than the static spin systems considered by Dhar, we recover his result for the noninteger dimensionality of the truncated- s -simplex lattices of coordination number s :

$$\epsilon = \frac{2\ln(s)}{\ln(2+s)}. \quad (1.2)$$

The scaling property of the Coleman-Weinberg potential, and the obtained values of ϵ , depend crucially upon the recursive construction of the lattices.

When we go over to theories involving fermions and Yang-Mills fields there are additional subtleties. We describe these qualitatively in Section 4. In the Conclusions we will address the question of physical interpretation.

2 Transformations for Deconstructed Lattice Scalars in $1+4$

We begin by considering transformations which augment or thin the degrees of freedom of $1+3$ theories of many complex scalar fields. These transformations stem from symmetries noticed long ago in the Ising model, [7, 8], (see the paper of Fisher [9] and references

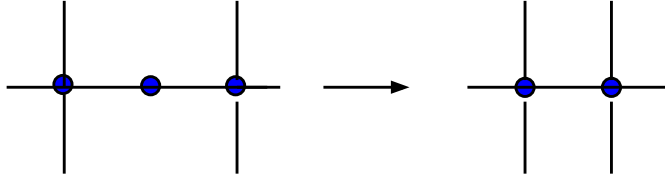


Figure 2: The 3-chain \rightarrow 2-chain dedecoration transformation integrates out the internal field and renormalizes the endpoint fields' kinetic terms and mass terms.

therein). In the language of Ising models a single spin_1 -link- spin_2 combination in the Hamiltonian can always be “decorated,” *i.e.*, written as a spin_1 -link- spin' -link- spin_2 interaction. That is, we can “integrate in” the new spin' , or “decorate” the original single link. Thus, an N -spin system can be viewed as a $2N$ spin system upon decorating. The decorations can be arbitrarily complicated, involving many new spins. Conversely, we can “integrate out” or “dedecorate” the spins internal to a chain whose endpoint spins are then renormalized (Fig.(2)).

Decoration is an exact transformation for Ising spins, and continuous spins in, (e.g., spherical models, which correspond to our models in the absence of kinetic terms). For us it is exact only in the limit of very large cut-off Λ . We are dealing with a transverse lattice [10] in which our “spins” are fields that have $1 + 3$ kinetic terms. We will perform decoration transformations truncating on quartic derivatives, such as $|\partial^2 \phi_a|^2 / \Lambda^2$. Nonetheless, it is a good approximate transformation in the $\Lambda \rightarrow \infty$ limit, or for the low lying states in the spectrum. The $1 + 3$ kinetic terms undergo renormalizations under these transformations, and thus distinguish the present construction from that of a spin model (e.g., the continuous complex spherical model) and these must be treated delicately.

We also require Onsager’s “star-triangle” or more generally, “polygon- \star ” transformations that replace a complete polygon of spins, Fig.(3), with a radiating star configuration, introducing a new central spin. This transformation can be done in field theory provided the plaquette is not oriented (which creates a complication when we attempt to include fermions and gauge fields). The polygon- \star transformations are, again, only exact for us in the $\Lambda \rightarrow \infty$ limit.

Combining sequences of polygon- \star and decoration transformations allows us to map a recursively defined lattice, such as the truncated s -simplex at k -th order into the same

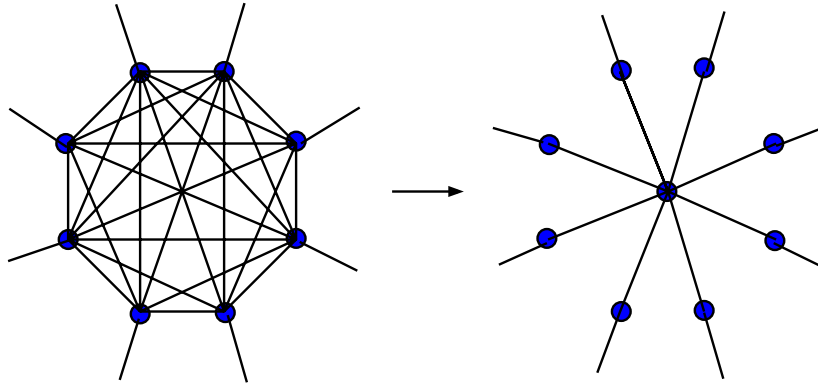


Figure 3: The polygon- \star transformation for a complete hexagon deletes the intrahexagonal links, and integrates in a new central field with radial links out the endpoint fields.

lattice at $k-1$ th order with different physical parameters. This sets up a renormalization group. The invariance of the Coleman-Weinberg potential in the large k limit allows us to determine the dimensionality of the theory. This limit corresponds to $\Lambda \rightarrow \infty$, so any corrections to the result after truncating the derivative expansion are expected to be small.

2.1 Decoration Transformations

2.1.1 3-chains \rightarrow 2-chains

We warm up with the simplest example of a decoration transformation applied to complex scalar fields. Consider an N complex scalar field Lagrangian in $1+3$, which can be viewed as a deconstructed S_1 compactified extra dimension with periodic boundary conditions:

$$\mathcal{L} = Z_0 \sum_{a=1}^N |\partial \phi_a|^2 - \Lambda_0^2 \sum_{a=1}^N |\phi_a - \phi_{a+1}|^2 - \mu_0^2 \sum_{a=1}^N |\phi_a|^2 \quad (2.3)$$

where we take N to be even and assume periodicity, hence $\phi_{N+a} = \phi_a$. It is convenient to allow for noncanonical normalization of the kinetic terms, and we thus display the arbitrary wave-function renormalization constant Z_0 .

It is useful to consider \mathcal{L} as a sum over 3-chains:

$$\mathcal{L} = \sum_{n \text{ odd}}^{N-1} \mathcal{L}_{n,n+2} \quad (2.4)$$

Each 3-chain involves three fields. The first 3-chain is:

$$\mathcal{L}_{1,3} = \frac{1}{2} Z_0 (|\partial \phi_1|^2 + 2|\partial \phi_2|^2 + |\partial \phi_3|^2) - \Lambda_0^2 |\phi_1 - \phi_2|^2 - \Lambda_0^2 |\phi_2 - \phi_3|^2 - \frac{1}{2} \mu_0^2 (|\phi_1|^2 + 2|\phi_2|^2 + |\phi_3|^2) \quad (2.5)$$

The fields ϕ_1 and ϕ_3 share half their kinetic terms and μ^2 terms with the adjacent chains, thus carry the normalization factors of $1/2$ within the chain (more generally, the endpoint fields may have $s - 1$ links with other fields and thus carry $1/s$ factors in the kinetic and mass terms of each chain). ϕ_2 can be viewed as a “decoration” of the chain. We can integrate out the internal field ϕ_2 and obtain an equivalent renormalized chain. Integrating out ϕ_2 :

$$\mathcal{L}_{1,3} = \frac{1}{2}Z_0(|\partial\phi_1|^2 + |\partial\phi_3|^2) - (\Lambda_0^2 + \frac{1}{2}\mu_0^2)(|\phi_1|^2 + |\phi_3|^2) + \Lambda_0^4(\phi_1 + \phi_3)^\dagger \frac{1}{Z_0\partial^2 + 2\Lambda_0^2 + \mu_0^2}(\phi_1 + \phi_3) \quad (2.6)$$

Expanding in the derivatives and regrouping terms gives:

$$\mathcal{L}_{1,3} = \frac{1}{2}Z_1(|\partial\phi_1|^2 + |\partial\phi_3|^2) - \Lambda_1^2|\phi_1 - \phi_3|^2 - \frac{1}{2}\mu_1^2(|\phi_1|^2 + |\phi_3|^2) - \delta_1|\partial(\phi_1 - \phi_3)|^2 + \mathcal{O}(\partial^4/\Lambda^2) \quad (2.7)$$

where we obtain:

$$\begin{aligned} Z_1 &= Z_0 \frac{8\Lambda_0^4 + 4\Lambda_0^2\mu_0^2 + \mu_0^4}{4\Lambda_0^4 + 4\Lambda_0^2\mu_0^2 + \mu_0^4} \approx 2Z_0 \\ \Lambda_1^2 &= \frac{\Lambda_0^4}{2\Lambda_0^2 + \mu_0^2} \approx \frac{1}{2}\Lambda_0^2 \\ \mu_1^2 &= \frac{2\mu_0^2\Lambda_0^2 + \mu_0^4}{2\Lambda_0^2 + \mu_0^2} \approx \mu_0^2 \\ \delta_1 &= \frac{Z_0\Lambda_0^4}{(2\Lambda_0^2 + \mu_0^2)^2} \approx \frac{1}{4}Z_0 \end{aligned} \quad (2.8)$$

We have written the approximate forms of the renormalizations of the parameters in the large Λ limit. Note that the μ^2 term is multiplicatively renormalized. This owes to the fact that it is the true scale-breaking term in the theory when the lattice is taken very fine, and Λ terms become derivatives. Since it alone breaks the symmetry of scale-invariance, it is therefore multiplicatively renormalized in free field theory (we have no interactions presently to produce anomalous dimension effects or additive scale breaking corrections).

The δ term has been written in the indicated form because, though it superficially appears to be a relevant $d = 4$ operator, it too is a quartic derivative on the lattice, *i.e.*, $(\partial^2 \text{ in } 1 + 3) \times (\text{a nearest neighbor hopping term on the lattice})$. It effects only the high mass limit of the KK mode spectrum. It is therefore dropped for consistency with the expansion to order ∂^4/Λ^2 .

The fields develop a new wave-function renormalization constant Z_1 . Note that in the $\Lambda \gg \mu$ limit, $Z_1 \rightarrow 2Z_0$, *twice* the original normalization. This renormalization is

common to all the ϕ_a fields in the other chains. Thus, we can write the original theory with N fields as a sum over the renormalized 2-chains, containing a total of $N/2$ fields:

$$\mathcal{L}' = Z_1 \sum_{a=1}^{N/2} |\partial\phi_a|^2 - \Lambda_1^2 \sum_{a=1}^{N/2} |\phi_a - \phi_{a+1}|^2 - \mu_1^2 \sum_{a=1}^{N/2} |\phi_a|^2 \quad (2.9)$$

The low energy spectrum of eq.(2.9) is identical to that of eq.(2.3). Moreover, if we take the limit $\Lambda \rightarrow \infty$ and $N \rightarrow \infty$, holding $M = \Lambda/N$ fixed, the spectrum becomes a ladder of KK-modes and is identical to that of eq.(2.3) for the first $N/2$ levels.

2.1.2 Renormalized 4-chains \rightarrow 2-chains

We will require in our applications presently the reduction of 4-chains, which are two endpoint fields and 2 internal decorating fields. We must allow for a more general parameterization of the chain fields. Generally, after performing polygon- \star transformations on our lattice, the full Lagrangian will be a sum over 4-chains:

$$\mathcal{L}^{full} = \sum_n \mathcal{L}_n^{4-chain} \quad (2.10)$$

These 4-chains will live on lattices with a coordination number s and generally have different normalizations for the two endpoint fields than the two internal fields.

Consider a typical 4-chain of the form:

$$\begin{aligned} \mathcal{L}^{4-chain} = & \frac{1}{s} Z_\Phi |\partial\Phi_1|^2 + Z_\phi |\partial\phi_1|^2 + Z_\phi |\partial\phi_2|^2 + \frac{1}{s} Z_\Phi |\partial\Phi_2|^2 \\ & - \frac{1}{s} \mu_0^2 (|\Phi_1|^2 + s|\phi_1|^2 + s|\phi_2|^2 + |\Phi_2|^2) \\ & - \Lambda_0'^2 |\Phi_1 - \phi_1|^2 - \Lambda_0^2 |\phi_1 - \phi_2|^2 - \Lambda_0'^2 |\Phi_2 - \phi_2|^2 \end{aligned} \quad (2.11)$$

We assume here that the endpoint Φ_i fields are shared with $s-1$ other neighboring chains, hence the Z_Φ/s kinetic term normalization, and the μ_0^2/s factors. Furthermore, note the central link for the internal fields, $|\phi_1 - \phi_2|^2$, has a different strength $\Lambda_0^2 \neq \Lambda_0'^2$ than the extremity links.

We integrate out the internal scalars ϕ_1 and ϕ_2 . This requires diagonalizing the ϕ_1 - ϕ_2 internal mass² matrix, which has eigenvalues $\Lambda_0'^2$ and $\Lambda_0'^2 + 2\Lambda_0^2$. We then regroup the derivative terms as before, discarding quartic and higher derivatives. We thus obtain a renormalized 2-chain:

$$\mathcal{L}^{2-chain} = \frac{\tilde{Z}}{s} (|\partial\Phi_1|^2 + |\partial\Phi_2|^2) - \tilde{\Lambda}^2 |\Phi_1 - \Phi_2|^2 - \frac{\tilde{\mu}^2}{s} (|\Phi_1|^2 + |\Phi_2|^2) + \mathcal{O}(\partial^4/M^2) \quad (2.12)$$

where:

$$\begin{aligned}
\tilde{Z} &= Z_\Phi + sZ\Lambda_0'^4 \left[\frac{1}{(\Lambda_0'^2 + \mu_0^2)^2} \right] \\
\tilde{\Lambda}^2 &= \frac{1}{2}\Lambda_0'^4 \left[\frac{1}{(\Lambda_0'^2 + \mu_0^2)} - \frac{1}{(\Lambda_0'^2 + 2\Lambda_0^2 + \mu_0^2)} \right] \\
\tilde{\mu}^2 &= \mu_0^2 \frac{(1+s)\Lambda_0'^2 + \mu_0^2}{\Lambda_0'^2 + \mu_0^2}
\end{aligned} \tag{2.13}$$

It is useful to define the ratio $\kappa = \Lambda_0'^2/\Lambda_0^2$ and consider the large Λ limit of these expressions:

$$\begin{aligned}
\tilde{Z} &\rightarrow Z_\Phi + Zs \\
\tilde{\Lambda}^2 &= \Lambda_0'^2 \frac{\kappa}{(\kappa + 2)} \\
\tilde{\mu}^2 &= \mu_0^2 (1 + s)
\end{aligned} \tag{2.14}$$

We will find that 4-chains arising after polygon- \star transformations on the truncated s -simplex lattices will have $\kappa = s$.

The full Lagrangian after replacing the 4-chains by the 2-chains and summing over all 2-chains, will take the form:

$$\mathcal{L}^{full} = \tilde{Z} \sum_a |\partial\Phi_a|^2 - \tilde{\mu}^2 \sum_a |\Phi_a|^2 - \tilde{\Lambda}^2 \sum_{a,b} |\Phi_a - \Phi_b|^2 \tag{2.15}$$

Note that when the 2-chains are summed, the $1/s$ factors disappear in overall kinetic and mass term normalizations.

2.2 Polygon- \star Transformations

Let us consider a “complete” deconstructed Lagrangian for a polygon of s sites. This is a highly nonlocal structure of Fig.(3) in which all sites are linked to all other sites with a common bond strength:

$$\mathcal{L}^{polygon} = Z_0 \sum_{a=1}^s |\partial\phi_a|^2 - \frac{1}{2}\Lambda_0^2 \sum_{a=1}^s \sum_{b=1}^s |\phi_a - \phi_b|^2 - \mu_0^2 \sum_{a=1}^s |\phi_a|^2 \tag{2.16}$$

Note that we must be careful not to double count, the link $|\phi_a - \phi_b|^2$ in double sums, hence the factors of $1/2$. It is interesting to compute the mass spectrum of the perfect polygon by itself, going into the Fourier basis:

$$\phi_a = \frac{1}{\sqrt{s}} \sum_{k=0}^{s-1} e^{\pi i k a / s} \chi_n; \quad \phi_{a+s} = \phi_a \tag{2.17}$$

whence:

$$\mathcal{L} = Z_0 \sum_{k=0}^{s-1} |\partial \chi_k|^2 - s \Lambda_0^2 \sum_{k=1}^{s-1} |\chi_k|^2 - \mu_0^2 \sum_{k=0}^{s-1} |\chi_k|^2 \quad (2.18)$$

Note the sum in the second term begins at $k = 1$, so the mode $k = 0$ is a zero mode when $\mu = 0$. Hence, renormalizing $\Lambda^2 = \Lambda_0^2/Z_0$ and $\mu^2 = \mu_0^2/Z_0$, the spectrum, consists of $s - 1$ degenerate modes of mass $\sqrt{s\Lambda^2 + \mu^2}$, and the single mode of mass μ with $k = 0$.

The polygon of s complex scalar fields admits a transformation which introduces a central complex scalar field Φ and becomes the s -star with $(s + 1)$ complex scalar fields. Let us consider the \star action in the form:

$$\mathcal{L}^\star = Z_\Phi |\partial \Phi|^2 + Z \sum_{a=1}^s |\partial \phi_a|^2 - \Lambda'^2 \sum_{a=1}^s |\Phi - \phi_a|^2 - \mu_\Phi^2 |\Phi|^2 - \mu'^2 \sum_{a=1}^s |\phi_a|^2 \quad (2.19)$$

Note that all $|\phi_a - \phi_b|^2$ bonds have been deleted and we introduce new $|\Phi - \phi_a|^2$ bonds radiating from the central scalar Φ .

Starting with \mathcal{L}^\star , we integrate out Φ :

$$\begin{aligned} \mathcal{L}^\star = & Z \sum_a^s |\partial \phi_a|^2 - (\Lambda'^2 + \mu'^2) \sum_a^s |\phi_a|^2 \\ & + \left[\Lambda'^4 \sum_{a,b}^s \phi_a^\dagger \frac{1}{Z_\Phi \partial^2 + s\Lambda'^2 + \mu_\Phi^2} \phi_b + h.c. \right] \end{aligned} \quad (2.20)$$

Performing the derivative expansion and reorganizing terms, we thus recover the polygon form of the Lagrangian:

$$\mathcal{L}^\star \rightarrow \mathcal{L}^{polygon} = Z_0 \sum_a^s |\partial \phi_a|^2 - \frac{1}{2} \Lambda_0^2 \sum_{a,b}^s |\phi_a - \phi_b|^2 - \mu_0^2 \sum_a^s |\phi_a|^2 + \mathcal{O}\left(\frac{\partial^2}{\Lambda^2}\right) \quad (2.21)$$

and we have the relations:

$$\begin{aligned} Z_0 &= Z + Z_\Phi \frac{s\Lambda'^4}{(s\Lambda'^2 + \mu_\Phi^2)^2} \approx Z + \frac{Z_\Phi}{s} \\ \Lambda_0^2 &= \frac{\Lambda'^4}{(s\Lambda'^2 + \mu_\Phi^2)} \approx \frac{\Lambda'^2}{s} \\ \mu_0^2 &= \frac{(\mu_\Phi^2 + s\mu'^2)\Lambda'^2 + \mu_\Phi^2 \mu'^2}{(s\Lambda'^2 + \mu_\Phi^2)} \approx \mu'^2 \frac{(s+1)}{s} \end{aligned} \quad (2.22)$$

where the approximate expressions hold in the large Λ limit, and are all that we ultimately require to implement the renormalization group.

Note that we have freedom within the \mathcal{L}^\star Lagrangian to vary the ratios Z_Φ/Z and μ_Φ/μ' . We can for example, choose $Z_\Phi = 0$, in which case Φ is a nonpropagating dummy field. The Φ field will recover a kinetic term when subsequent chain transformations are performed. The $Z_\Phi = 0$ case is interesting for Yang-Mills, and corresponds to “integrating in” an infinite coupling constant gauge field, and the infinite coupling will run to a finite value after subsequent chain transformations. Presently we will make the convenient choice $\mu'^2 = \mu_\Phi^2$, but we do not specify explicitly Z_Φ/Z . This will act as a check on our result.

We can readily invert the transformation in the large Λ limit. In summary, the polygon Lagrangian:

$$\mathcal{L}^{polygon} = Z_0 \sum_{a=1}^s |\partial\phi_a|^2 - \frac{1}{2}\Lambda_0^2 \sum_{a=1}^s \sum_{b=1}^s |\phi_a - \phi_b|^2 - \mu_0^2 \sum_{a=1}^s |\phi_a|^2 \quad (2.23)$$

can be replaced by the \star Lagrangian:

$$\mathcal{L}^\star = Z_\Phi |\partial\Phi|^2 + Z \sum_{a=1}^s |\partial\phi_a|^2 - \Lambda'^2 \sum_{a=1}^s |\Phi - \phi_a|^2 - \mu'^2 |\Phi|^2 - \mu'^2 \sum_{a=1}^s |\phi_a|^2 \quad (2.24)$$

with the choice of parameters ($\Lambda \rightarrow \infty$):

$$\begin{aligned} \frac{Z_\Phi}{s} + Z &= Z_0 \\ \Lambda'^2 &= s\Lambda_0^2 \\ \mu'^2 &= \mu_0^2 \frac{s}{s+1} \end{aligned} \quad (2.25)$$

2.3 Combining Polygon- \star and 4-chain Transformations to Reduce the Truncated s -simplex Lattice

We consider now any k th order s -simplex lattice built up recursively as described in Section I. The lattice has $N_k = (s+1)s^k$ complex scalar fields and $L_k = (s+1)s^{k+1}/2$ links, where each site has coordination number s . For concreteness consider Fig.(4A), the second order 3-simplex.

The Lagrangian takes the form:

$$\mathcal{L}_k = Z_0 \sum_{a=1}^{N_k} |\partial\phi_a|^2 - \Lambda_0^2 \sum_{links} |\phi_a - \phi_b|^2 - \mu_0^2 \sum_{a=1}^{N_k} |\phi_a|^2 \quad (2.26)$$

We begin by performing the polygon- \star transformations on each of the elementary polygons. All of the elementary polygons are annihilated by this procedure, replaced by stars,

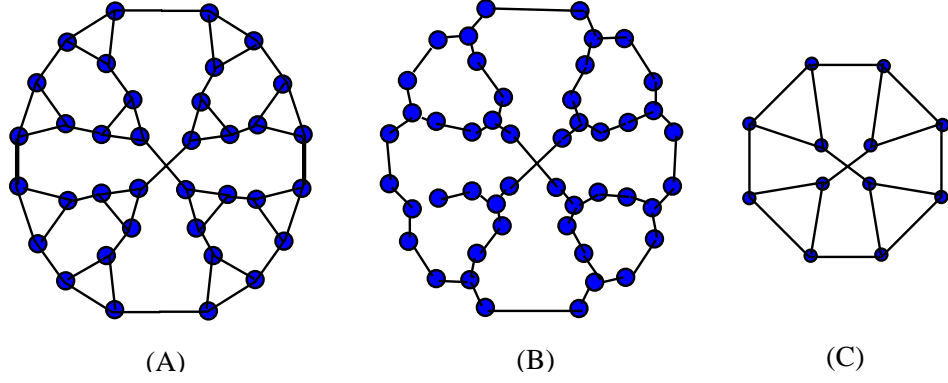


Figure 4: Illustration of $k \rightarrow k - 1$ RG transformation. (A) Second (k th) order truncated 3-simplex; (B) reduced after polygon(triangle)- \star transformations; (C) reduced to first ($k - 1$ th) order after 4-chain \rightarrow 2-chain transformations.

and the lattice of Fig.(4A) is carried into that of Fig.(4B). The centers of the stars are connected to neighbors through 4-chains, and the full Lagrangian is now a sum over 4-chains. We have the 4-chain parameters determined by eq.(2.25):

$$\begin{aligned} \frac{Z_\Phi}{s} + Z &= Z_0 \\ \Lambda'^2 &= s\Lambda_0^2 \\ \mu'^2 &= \mu_0^2 \frac{s}{s+1} \end{aligned} \quad (2.27)$$

Now we reduce the 4-chains to 2-chains. The lattice is mapped from Fig(4B) into Fig.(4C). We see that we have now reduced the original k th order lattice to the $k - 1$ th order with the new Lagrangian:

$$\mathcal{L}_{k-1} = \tilde{Z} \sum_{a=1}^{N_{k-1}} |\partial\phi_a|^2 - \Lambda^2 \sum_{links}^{L_{k-1}} |\phi_a - \phi_b|^2 - \mu^2 \sum_{a=1}^{N_{k-1}} |\phi_a|^2 \quad (2.28)$$

The parameters renormalize as in eq.(2.14) where $\kappa = \Lambda'^2/\Lambda^2 = s$. The resulting overall normalization is:

$$\begin{aligned} \tilde{Z} &= sZ_0 \\ \tilde{\Lambda}^2 &= \Lambda_0^2 \frac{s}{(s+2)} \\ \tilde{\mu}^2 &= \mu_0^2 s \\ \tilde{N} \equiv N_{k-1} &= \frac{N_k}{s} \end{aligned} \quad (2.29)$$

We have also noted the change in the number of fields, N_k . We see that the arbitrariness of choosing Z_Φ/Z (also, μ_Φ/μ' , which we fixed to unity) in the intermediate step is hidden, a symmetry of the result.

3 Computation of the Dimensionality

3.1 Vacuum Energy Scaling Law

We have described a theory of free complex scalars defined on the k th iteration of the kernel lattice. For the truncated- s -simplex lattices the coordination number is s , the number of fields in the k th iteration is $N_k = (s+1)(s)^k$ and the number of links is $L_k = (s+1)s^{k+1}/2$. The Lagrangian is:

$$\mathcal{L} = Z \sum_{a=1}^{N_k} |\partial \phi_a|^2 - \Lambda^2 \sum_{a,b}^{L_k} |\phi_a - \phi_b|^2 - \mu^2 \sum_{a=1}^{N_k} |\phi_a|^2 \quad (3.30)$$

where the linking mass term sums over the links.

If we could Fourier transform eq.(3.30) we would obtain a mass spectrum of the form $M_n^2 = \omega_n^2 + \mu^2$. The path integral for our theory then takes the form in a Euclidean momentum space, up to an overall multiplicative normalization factor:

$$e^{-\Gamma} = \int D\phi e^{-\int d^4x \mathcal{L}} = \prod_{p_\mu} \prod_{n=1}^{N_k} (Zp^2 + \omega_n^2 + \mu^2)^{-1} \quad (3.31)$$

The vacuum energy, or Coleman-Weinberg potential, is up to an overall additive constant:

$$\Gamma = Z^{-2} \int \frac{d^4p}{(2\pi)^4} \sum_n \ln(p^2 + \omega_n^2 + \mu^2) \quad (3.32)$$

where we have rescaled the 4 momentum integral by Z .

We want to replace the sum on n by a continuous momentum integral. In ϵ dimensions we interpret n as a radial coordinate and replace:

$$\sum_{n=0}^{N_k} \rightarrow \int_0^{N_k} dn (n/N_k)^{\epsilon-1} (\epsilon \Omega_\epsilon) \quad (3.33)$$

where Ω_ϵ is the solid angle in ϵ dimensions. With n a radial coordinate, the leading behavior at low n of ω_n^2 is $\omega_n^2 \sim c(n/N_k)^2 \Lambda^2$, where c is a constant.

Let us rescale n to write the integral over an $4 + \epsilon$ dimensional momentum vector,

$$p^2 = p_\mu p^\mu + c(n/N_k)^2 \Lambda^2 \quad (3.34)$$

The Coleman-Weinberg potential becomes:

$$\Gamma = c(\epsilon) \frac{Z^{-2} N_k}{\Lambda^\epsilon} \int \frac{d^{4+\epsilon}p}{(2\pi)^{4+\epsilon}} \ln(p^2 + \mu^2) \quad (3.35)$$

where $c(\epsilon)$ is an overall normalizing factor coming from the c dependence, solid angle, and other factors etc., that are irrelevant for the scaling argument. The integral, apart from the explicit scaling prefactor, is finite for nonzero ϵ . It is thus insensitive to k as $k \rightarrow \infty$ to its UV cut-off limit and to $\Lambda \rightarrow \infty$.

We can thus scale the mass parameter μ out of the integral to write:

$$\Gamma = \frac{Z^{-2} N_k}{\Lambda^\epsilon} (\mu)^{4+\epsilon} V(\epsilon) \quad (3.36)$$

where $V(\epsilon)$ is insensitive to k as $k \rightarrow \infty$.

Thus, suppose we know the value of the parameters Z , μ , Λ and N_k for some large value of k . Then, we obtain the Coleman-Weinberg potential for $k-1$ by the sequence of RG transformations and we find new parameters:

$$\tilde{Z} = h(s)Z, \quad \tilde{\mu} = f(s)\mu, \quad \tilde{\Lambda} = g(s)\Lambda \quad N_{k-1} = N_k/s. \quad (3.37)$$

Since V is insensitive to k for large k (and large Λ) we have:

$$Z^{-2} \frac{N_k}{\Lambda^\epsilon} (\mu)^{4+\epsilon} = \frac{(\tilde{Z})^{-2} N_{k-1}}{(\tilde{\Lambda})^\epsilon} (\tilde{\mu})^{4+\epsilon} \quad (3.38)$$

or

$$1 = \frac{(f(s))^{4+\epsilon}}{s(h(s))^2(g(s))^\epsilon} \quad (3.39)$$

and thus the dimensionality is determined as:

$$\epsilon = \frac{(-4 \ln(f(s)) + \ln(s) + 2 \ln(h(s)))}{\ln(f(s)/g(s))} \quad (3.40)$$

3.2 Dimensionality and RG Invariants of the Truncated- s -simplex Lattices

Let us compute the dimensionality of the truncated s -simplex lattices. From eqs.(2.29) we have:

$$h(s) = s, \quad f(s) = \sqrt{s}, \quad g(s) = \sqrt{\frac{s}{s+2}}. \quad (3.41)$$

The dimensionality is thus:

$$\epsilon = \frac{2 \ln(s)}{\ln(s+2)} \quad (3.42)$$

We have thus recovered the result of Dhar for the dimensionality of spin-systems on the truncated s -simplex lattices. In Dhar's analysis of spins systems, the spins are static, *i.e.*,

have no kinetic terms in an auxiliary 1+3 dimensions. The wave-function renormalizations are essential in our present renormalization group and to the scaling law for the Coleman-Weinberg potential. Nonetheless, the lattice dimensionality is the same as in the static spin system.

Note that the coordination number must satisfy $s > 2$ for a nontrivial noninteger dimensionality. For $s = 2$ the dimension is always 1. For the truncated s -simplex lattices we have $1 \leq \epsilon \leq 2$.

The scaling laws amongst the four quantities of eqs.(2.29) imply that there are 3 invariants. We have just encountered one invariant from the vacuum energy scaling law, and we see that there are two others:

$$\begin{aligned} V_r &= \frac{(\tilde{Z})^{-2} N_{k-1} (\tilde{\mu})^{4+\epsilon}}{(\tilde{\Lambda})^\epsilon} \\ \mu_r^2 &= \frac{\tilde{\mu}^2}{\tilde{Z}} \\ N_r &= \tilde{Z} \tilde{N} \end{aligned} \tag{3.43}$$

It is useful to define a noninvariant renormalized cut-off:

$$\Lambda_r^2 = \frac{\tilde{\Lambda}^2}{\tilde{Z}} \tag{3.44}$$

Λ_r can be used as the “running” mass scale. Combining with the the RG invariant vacuum energy scaling factor V_r allows us to define a RG invariant mass scale:

$$M = \frac{\tilde{\Lambda}_r}{(\tilde{N})^{1/\epsilon}} \tag{3.45}$$

The scale M is fixed in the large Λ_r and \tilde{N} limit, and has nothing to do with the physical mass $\tilde{\mu}/\sqrt{\tilde{Z}}$. It defines the threshold scale of the KK-modes, *i.e.*, the effective compactification scale of the theory. The number of KK modes with energy E is given by

$$n(E) = \left(\frac{E}{M} \right)^\epsilon \tag{3.46}$$

The physical significance of the invariance of $\tilde{Z} \tilde{N}$ pertains to interacting theories, such as Yang-Mills gauge theories. We can identify:

$$\tilde{Z} \propto \frac{1}{g^2} \tag{3.47}$$

a common dimensionless coupling constant of the deconstructed theory defined at the scale of the cut-off. Then the invariant M tells us how the coupling constant scales with

choice of cut-off. To see the running coupling constant scaling law combine eq.(3.45) and eq.(3.47) to obtain:

$$\frac{1}{g^2(\Lambda_r)} \propto \left(\frac{M}{\Lambda_r}\right)^\epsilon \quad (3.48)$$

Thus, we recover normal power-law running of $g^2(\Lambda_r)$ when ϵ takes on integer values. The formula exhibits the generalization for noninteger dimension.

4 Considerations of Gauge Fields and Fermions

Naturally, we are interested in realistic models built along the lines suggested here. Thus we will require Yang-Mills, and fermions, including chirality. The present discussion will be qualitative, as we note some new issues that arise in attempting this extension.

When we go over to theories involving fermions and Yang-Mills fields there are additional subtleties. These subtleties revolve around the polygon- \star transformation. For example, Wilson fermions in a polygon cannot be mapped to the \star configuration. Similarly, the PNGB's of Yang-Mills theories that are periodically compactified must be lifted by plaquettes in order to perform the polygon- \star transformation. The point is that the polygon is orientable, while the \star is not, so orientational elements of the action will not be carried through by the transformation. In the Yang-Mills case, an arbitrary magnetic flux threading a plaquette, $\int B \cdot dA \sim \oint A \cdot dx$ cannot be represented by the \star form of the action, and this requires that a certain PNGB be infinitely heavy. The \star configuration, however, will be seen to be the key to creating chiral fermions. Chiral fermions in deconstruction are lattice defects. In the present case they must be incorporated as the centers of \star configurations that are invariant under the RG transformations used to reduce the lattice. In a sense then, chiral fermions are rarified defects, or invariant centers in the fractal lattice, similar to doping atoms in a material, or to the centers of snowflakes.

Yang-Mills gauge fields are introduced in a deconstructed theory by having gauge groups, G_a , living on sites and linking-Higgs-fields defined on links. We also include plaquette terms which show up as mass terms of PNGB's in the 1 + 3 dimensions. Hence, let us choose $G_a = SU(N)$ with a common coupling constant g , and the link field Φ_{ab} is then an (\overline{N}, N) chiral field with a VEV, $\langle \Phi \rangle = vI_N$. The Lagrangian is then:

$$\mathcal{L}_{YM} = - \sum_{n=1}^{N_k} \frac{1}{4g^2} G_{\mu\nu}^a G^{a\mu\nu} + \sum_{(ab)}^{L_k} \text{Tr}[D_\mu, \Phi_{ab}]^\dagger [D^\mu, \Phi_{ab}] + \sum_{\text{plaq } n}^{P_k} \lambda_n \text{Tr}[\prod_{\text{plaq } n} \Phi_{ab}] \quad (4.49)$$

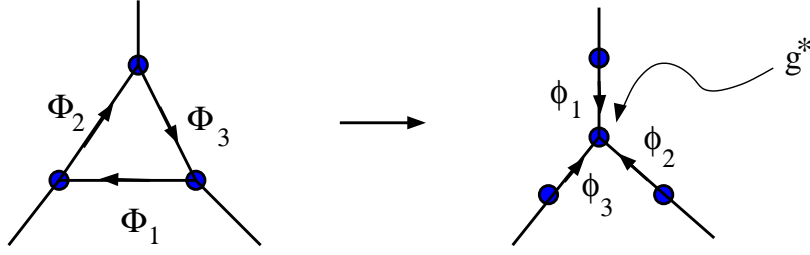


Figure 5: The triangle- \star transformation for an irreducible plaquette maps $\Phi_i \rightarrow \phi_j$, but imposes a constraint, $\Phi_3\Phi_2\Phi_1 = 1$.

The *irreducible* plaquettes are those which do not encircle a subplaquette (*i.e.*, can be contracted). The irreducible plaquettes are $P_k = (s+1)^{k+1}$.

\mathcal{L}_{YM} has been supplemented with a plaquette action, where each plaquette has a coupling constant λ_k . Let us first consider $\lambda_k = 0$. Then the theory will contain a spectrum of 1 vector zero mode, $N_k - 1$ massive gauge fields (KK-modes), and in tree approximation $L_k - N_k + 1$ massless PNGB's. The PNGB's will generally be lifted in perturbation theory to masses of order αM^2 , but they can also be elevated by turning on the λ_k . Indeed, we see that $P_k \gg L_k$, so including all irreducible plaquettes with large λ_n we can lift all PNGB's, except for a single zero-mode.

Lifting the PNGB's is necessary for the implementation of the \star -chain RG. In Fig.(5) we see a mapping of the irreducible triangle with link fields Φ_i into a star configuration with new link fields ϕ_j . The net gauge phase rotations in going from one site to another must be faithfully represented under this redefinition, thus:

$$\Phi_1 = \phi_3^\dagger \phi_2 \quad \Phi_2 = \phi_1^\dagger \phi_3 \quad \Phi_3 = \phi_2^\dagger \phi_1 \quad (4.50)$$

and we thus see that the Φ_i are constrained:

$$1 = \Phi_3\Phi_2\Phi_1 \quad (4.51)$$

This is the orientability problem mentioned above. It requires the quantization of the Wilson loop around the triangle plaquette $g \oint A_A dx^A = 2\pi n$ (more properly, $\Phi_3\Phi_2\Phi_1$ must lie in the center of the group). In the deconstruction language, it imposes a constraint on the PNGB's. We can treat this constraint by introducing terms $\sim \lambda_{123} \text{Tr}(\Phi_3\Phi_2\Phi_1)$ for all elementary plaquettes, and we treat λ_n as a Lagrange multiplier, then perform the polygon- \star transformation. This will lift the PNGB's from the spectrum. This is

the expected decoupling that of high mass PNGB's must occur when the short-distance degrees of freedom are thinned. Thus, we expect that polygon- \star transformations should make sense in the theory with plaquettes.

Another intriguing point is that the \star Lagrangian involves “integrating in” additional Yang-Mills gauge groups at the centers of stars with coupling constants g^\star . As we saw in the star transformation, there is a freedom to choose the wave-function renormalization constant, Z_Φ , arbitrarily relative to its neighbors. This translates into the freedom of choosing the coupling constant g^\star for the new central gauge group arbitrarily. In particular, we can choose $g^\star = \infty$, which completely suppresses the continuum kinetic term of the new gauge field at this scale. The subsequent chain transformations will induce a finite coupling and gauge invariant kinetic term for this gauge field as we perform the 4-chain \rightarrow 2-chain transformations. The renormalized couplings after the combined transformations for all gauge fields will have a common value and will run according to the scaling laws described in the previous section.

Barring therefore, particular topological obstructions, we expect that the reduction for Yang-Mills gauge fields should also go through in the Gaussian approximation, and we will obtain the same dimensionality as for complex scalars. Obviously the question of interactions is of great importance. We certainly expect that there are $1 + 3$ continuum renormalization group effects that accompany the lattice reduction, which corresponds to a change of scale (*e.g.*, of Λ_r). The main issue, however, comes from the power-law running in eq.(3.48). The Yang-Mills coupling constant as described will reach, evolving upward with scale, a unitarity bound, $g^2 \sim (4\pi)^2$ at an effective scale Λ_r^\star fairly quickly (it would be interesting to construct models in which $\epsilon \ll 1$ where the power law running is suppressed, and appears approximately logarithmic). This is the scale of unitarity breakdown for longitudinal KK-mode scattering [3]. It implies a phase transition in the theory, which is usually considered to be the string transition. Another logical possibility is that g^2 runs large, but then is “reset” to a small value by a dynamical transition in the theory, then runs large again, etc., leading to a *limit cycle*. With a limit cycle it may be possible to take $\Lambda_r \rightarrow \infty$ in the interacting theory as well, without a transition to the string phase.

Fermions pose additional challenges. Fermions live on sites and will have kinetic “hopping terms” on the links. We can always view the lattice as a fermion mass matrix, take all fermions to be vectorlike, and choose the hopping terms to be mass terms. This would readily admit polygon- \star transformations and RG reduction of the lattice as we

have derived. This would seem to us to be a relatively uninteresting case.

The hopping terms should be built out of γ matrices. We expect that we require the use of all γ matrices through $\gamma_{4+[2s]}$ in construction of the action. Hence, for $s = 2$, γ_5 suffices, such as in S_1 , while $s = 3$ requires γ_6 , etc. Consider the polygon of Fig.(5) for $s = 3$ with Yang-Mills and the fermionic kinetic terms hopping around the perimeter of the polygon. Using γ_5 , the kinetic terms will be hopping terms of the form $\sum_n v \bar{\psi}_n \gamma_5 \Phi_n \psi_{n+1}$. Generally this form leads to the fermion doubling problem [11], but appears to us again to admit polygon- \star transformations. It is most sensible to consider fermions with the Wilson term (the Wilson term structure will always occur with SUSY). This is discussed in detail in [11]. With the Wilson term the hopping terms are written as $\sum_n (v \bar{\psi}_{nL} \Phi_n \psi_{(n+1)R} - v \bar{\psi}_{nR} \psi_{nR})$ where the Dirac-like mass term on brane n is really part of the kinetic term (we can swap $L \leftrightarrow R$ by a parity redefinition [11]). The Wilson term has a definite orientational sense of $L(n) \rightarrow R(n+1)$ around the polygon. Hence, the fermionic kinetic terms in the presence of the Wilson term orient the polygon. These cannot be reduced by polygon- \star transformations. It is not clear to us that sensible reducible fermionic actions exist, but admitting the RG reduction is only a convenience in computing the dimensionality of the lattice. More exotic fermionic reductions that do not require the polygon- \star transformation may exist. Another possibility is that lattices involving γ_6 , γ_7 , etc. may exist for which nontrivial RG's exist. These are open questions.

If we use the “dumb” action with vectorlike fermions and mass matrix hopping terms we can still introduce chirality. We must construct “invariant stars” which are dislocations in the lattice and are not reduced by the RG transformations (it is not hard to convince oneself that truncated s -simplex lattices can be constructed in this way). At the center of the invariant star configuration we introduce a chiral fermion, Ψ_L . The fermion has radial hopping terms to the perimeter fermions of the form $\sum_n v \bar{\Psi}_L \phi_n \psi_{nR}$. By “doping” a dumb lattice with the appropriate number of chiral dislocations one evidently can make a fractal extension of the Standard Model.

5 Physical Interpretations and Conclusions

How do we interpret these new theories physically? Fractional extra dimensions are not really compactified extra dimensions, since no global boundary condition is introduced which corresponds to a global compactification. Indeed, the kernel lattice appears to be a matter theory with dimensionality 0. Rather, we introduce a scale, Λ , which may be

viewed as a cut-off and is our (inverse) lattice scale. We ultimately imagine $\Lambda \rightarrow \infty$. There are a large number of fields N and we take $N \rightarrow \infty$. The analogue of the compactification scale is the RG invariant M , as in the case of a regular lattice description of a continuous extra dimension, and is a physical scale held fixed in the limit.

There are thus two physical interpretations for these constructions. The first is an “outer” modification of spacetime. Here we have in mind a dimensional transition at the scale M in which we view the continuous $1 + 3$ dimensions as *a brane in a higher dimension with a surface structure* with characteristic scale length $1/M$. This brane surface is viewed as dynamical, and will have “brane surface chemistry,” analogous to surfaces in condensed matter physics, and may arise from an interface with an exterior region involving new physics. The fractal theory space is an effective description of such a system. The fractality, in analogy with surface layers on material media, may arise because the interface with the extra outside dimensions involves a region of rapid change in physical parameters. In this picture Lorentz invariance at short distances strictly only applies to the $1 + 3$ dimensions, but with Λ large the relevant low energy physics of the dimensional transition scale M is approximately Lorentz invariant in $1 + (3 + \epsilon)$ dimensions. In this case, the scale Λ cannot be infinite and may represent a further higher energy transition to string theory.

An alternative, and perhaps more intriguing view, is an “inner” modification of physics, in that the scale M represents a true dimensional transition to $1 + (3 + \epsilon)$ dimensions, enhancing Lorentz invariance. Then at all shorter distances the noninteger dimensionality is preserved. This is a remarkable possibility in that a quantum field theory defined in $1 + (3 + \epsilon)$ dimensions with irrational ϵ is finite to all orders in perturbation theory. The cut-off scale Λ can be taken to infinity with impunity, holding M fixed as the defining dimensional transition scale. The low energy physics is a fixed point under a renormalization

If we were naive, we would speculate that we have given a prescription for the construction of *finite* quantum field theories of matter. Thus, all infinities in $1 + 3$ dimensions of the Standard Model would be associated with the cut-off scale M , which is the threshold for new physics associated with the noninteger correction to the dimension of space-time. Above the scale M we treat the field theory with ‘t Hooft and Veltman’s dimensional regularization as the exact calculational tool for $4 + \epsilon$ dimensions. Thus, one way to treat the Standard Model as a quasi-noninteger dimensional theory would be to replace all loop

integrals in 4-dimensions by $4 + \epsilon$ dimensions above a fixed matching scale M :

$$\text{Loops} \rightarrow \int_0^M \frac{d^4 k}{(2\pi)^4} + \int_M^\infty \frac{d^{4+\epsilon} k}{(2\pi)^4} \quad (5.52)$$

Modulo ambiguities in treating γ^5 , This theory is apparently finite above M to all orders in perturbation theory with.

Unfortunately, the power-law running of the coupling constants, $g^2 \sim (\mu/M)^\epsilon$ with $\epsilon > 0$ implies that the theory undergoes a phase transition at a strong coupling scale $g^2 \sim (4\pi)^2$. If we could find sensible theory spaces with $\epsilon < 0$, the couplings would always be asymptotically free. Otherwise this cannot be a complete prescription for a finite theory, and still requires imbedding into something else, or develops a self-replicating limit cycle. In any case we must account for gravity, and imbedding into string theory would seem to be the most sensible option. This prescription is nonetheless worthy of study, and is a “continuous KK-mode distribution approximation” to any theory that envelops the Standard Model into a noninteger extra dimension ϵ . We infer that the Higgs boson will receive radiative corrections to its mass from top quark loops:

$$m_H^2 \sim -\frac{3g_t^2}{8\pi^2} M^2 \quad (5.53)$$

(or heavier fermions in an extension of the model). Hence, we infer that M^2 is of order $\sim \text{TeV}$ (a Little Higgs model can raise the scale to $\sim 10 \text{ TeV}$ through custodial chiral symmetries; SUSY may be relevant as well).

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